Advanced and algorithmic graph theory

Martina Tscheckl

February 29, 2016

Please send feedback to martina@tscheckl.eu.

Contents

1 Introduction and notations

29.02.2016

Let $G = (V, E)$ be a graph. Then V is the *vertex set* of G and E is the *edge set* of G. If

$$E \subseteq V \times V$$

then G is called a *directed graph*. And if

$$E \subseteq \{\{a, b\} : a \neq b, a, b \in V\}$$

then G is called an *undirected graph*.
A *trivial graph* is the empty graph $G = (\emptyset, \emptyset)$.

We will always consider the case of undirected graphs if not specified otherwise.

Definition (Order of G). *The* order of G *is denoted by* $|G| := |V|$. *We assume that* V *is finite if not otherwise specified. And we denote by* $\|G\| := |E|$.

Notation (Edges). *Edges are denoted by* $\{i, j\}$, (i, j), *or* ij. *If* $e = \{i, j\} \in E$, *then*

(a) i *and* j *are* adjacent,

(b) i *is* incident *to* e *(or* i *and* e *are incident),*

(c) i *and* j *are* neighbours.

Definition (Complete graph). *A graph* $G = (V, E)$ *is called a* complete graph *if and only if*

$$E = \{\{a, b\} : a \neq b, a, b \in V\}.$$

It is called K_n *if* $|V| = n$.

Definition (Independet or stable set). *A* set of vertices $A \subseteq V$ *is called* independent *or* stable *if and only if*

$$\forall a, b \in A : \{a, b\} \notin E$$

Definition (Isomorphic). *Two graphs* $G = (V, E)$ *and* $G' = (V', E')$ *are* isomorphic *if and only if there exists a bijective map* $\varphi : V \to V'$ *such that for all* $a, b \in V$

$$\{a, b\} \in E \iff \{\varphi(a), \varphi(b)\} \in E'.$$

Then φ *is called an* isomorphism *and we write* $G \equiv G'$.

Definition (Graph property). *A class of graphs that is closed under isomorphisms is called a* graph property.

Example (Triangle). *Let* $G = K_3$. *Then* $G' \equiv G$ *implies that* G' *is a triangle. Another example would be* K_4.

add pic

Definition (Graph invariant). *A mapping taking graphs as arguments is called a graph invariant if and only if it assigns equal images (values) to isomorphic graphs.*

Examples. 1. *Number of vertices,*

2. *Number of edges,*

3. *Longest number (cardinality of longest clique) of pairwise adjacent vertices.*

Definition (Clique)**.** *A subset $A \subseteq V$ is called a* clique *if and only if*

$$\forall a, b \in A,\ a \neq b \implies \{a, b\} \in E.$$

Definition (Union and intersection of graphs)**.** *Let G and G' be two graphs. Then we define*

1. *the* union *of two graphs as*

$$G \cup G' := (V \cup V', E \cup E')$$

2. *the* intersectoin *of two graphs as*

$$G \cap G' := (V \cap V', E \cap E')$$

If $G \cap G' = (\emptyset, \emptyset)$, we say G and G' are disjoint*.*

Definition (Subgraphs)**.** 1. *If $V \subseteq V'$ and $E \subseteq E'$, we say G is* subgraph *of G' and write $G \leq G'$.*

2. *If $G \leq G'$ and $G \neq G'$, we say G is a* proper subgraph *of G'.*

3. *If $G \subseteq G'$ such that*

$$\forall a, b \in V(G) : \{a, b\} \in E' \implies \{a, b\} \in E,$$

then G is an induced subgraph*. We say $V := V(G)$* induces *or* spans *G in G' and denote it by $G'[V]$.* add pic

Definition (Adding/removing vertices or edges in/from graphs)**.** *Let $G = (V, E)$ and $G' = (V', E')$ be graphs.*

(a) *If $U \subseteq V(G)$, we write*

$$G - U := G[V \setminus U].$$

If $U = \{v\}$, we write $G - v$ instead of $G - \{v\}$. add pic

(b) *If $G' \subseteq G$, we write $G - G' := G - V(G')$* add pic

(c) *If $F \subseteq E$, we write*

$$G + F := (V, E \cup F)$$

and

$$G - F := (V, E \setminus F).$$

If $F = \{e\}$, we write $G + e$ instead of $G + \{e\}$ and $G - e$ instead of $G - \{e\}$.

Definition (Edge maximal with respect to a given graph property)**.** *A graph G is called* edge maximal with respect to a given graph property *if and only if G itself has the property, but no graph*

$$G + \{x, y\}$$

has the property for some $x, y \in V(G)$, $x \neq y$ with $\{x, y\} \notin E(G)$.

Example. *Let G be a graph with property P, where P = "triangle free".*

(a) add pic

(b) add same pic

Both graphs are maximal with respect to P.

Remark. *If we call a graph* minimal or maximal with respect to some property *without any other specification of the order, then it is meant to be according to the subgraph relation.*

Definition (Product of graphs)**.** *If G and G' are disjoint, define $G * G'$ as a graph obtained from*

$$G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$$

by adding all edges $\{x, y\}$ with $x \in V(G)$ and $y \in V(G')$. add pic

Definition (Complement graph)**.** *The* complement *of G is denoted by G^C or $\overline{G}$ and is defined as*

$$\overline{G} := (V(G), \{\{a, b\} : a \neq b,\ a, b \in V(G)\} \setminus E(G))$$

Definition (Line graph)**.** *The line graph of G is denoted by*

$$L(G) = (E(G), \{\{e, f\} : e, f \in E, e \neq f, e \cap f \neq \emptyset\})$$

add pic 7

Definition (Degree of G)**.** *Denote the set of neighbours of a vertex $v \in V$ by $N_G(v)$. Then we define* $\deg_G(v) \equiv d_G(v) := |N_G(v)|$ *as the* degree *of v in G. If $d_G(v) = 0$ we say that v is* isolated *in G.*
We define

1. *the* minimum degree of G *as*

$$\delta(G) = \min_{v \in V(G)} d_G(v)$$

2. *the* maximum degree of G *as*

$$\Delta(G) = \max_{v \in V(G)} d_G(v)$$

3. *the* average degree of G *as*

$$d(G) = \frac{1}{|V(G)|} \sum d_G(v)$$

Definition (k-regular graph)**.** *A graph G is k-regular if and only if*

$$\deg_G(v) = k$$

for all $v \in V$ and for some $k \in \mathbb{N}_*$.
If $k = 3$, we call G cubic.
We define

$$\varepsilon(G) \coloneqq \frac{|E|}{|V|}.$$

Definition (Path)**.** *A path is a nonempty graph $P = (V, E)$ of the form*

$$V = \{x_0, x_1, \dots, x_k\}$$

and

$$E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$$

where all edges are all pairwise distinct. The vertices x_0 and x_k are the end vertices *of P. And the vertices x_i for $1 \leq i \leq k-1$ are the inner vertices of P.*

Definition (Length of path)**.** *Let $P = (V, E)$ be a path. The* length of the path *is defined as the number of edges $|E|$. A path of length k is denoted by P^k. (Notice that $k = 0$ is possible)*

add pic

Remark. *We often refer to a path P^k as $x_0x_1 \dots x_k = P^k$.*

Notation. *Let $P = x_0x_1 \dots x_k$. We write*

$$Px_i \coloneqq x_0 \dots x_i$$
$$x_iP \coloneqq x_i \dots x_k$$
$$x_iPx_j \coloneqq x_i \dots x_j$$

Let $\mathring{P} \coloneqq x_1x_2 \dots x_{k-1}$. Then we write

$$\mathring{P}x_i \coloneqq x_0 \dots x_{i-1}$$
$$x_i\mathring{P} \coloneqq x_{i+1} \dots x_k$$
$$x_i\mathring{P}x_j \coloneqq x_{i+1} \dots x_{j-1} \equiv x_{i+1}Px_{j-1} \text{ for } i+1 \leq j$$

add pic

Definition (A-B-path)**.** *Let $A, B \subseteq V(G)$. A path $P = x_0x_1 \dots x_k$ is callled an A-B-*path *if*

$$V(P) \cap A = \{x_0\}$$

and

$$V(P) \cap B = \{x_k\}.$$

*If $A = \{a\}$ and $B = \{b\}$ write a-b-*path *instead of $\{a\}$-$\{b\}$-path.*

add pic

Definition (Independent path)**.** *Two or more paths are* independent *if and only if none of them contains as inner vertex an inner vertex of some other path.*

Example. *The paths $P_1 = x_0x_1x_2x_3$ and $P_2 = y_0y_1y_2y_3$ are independent. If the path $P_3 = x_0x_2y_2$ is added, they are not an independent set of paths anymore.* add pic

Definition (H-path)**.** *Let H be a given graph. We call a path P an H-path if P is non-trivial and*

$$V(P) \cap V(H) = \{x_0, x_k\}$$

where x_0 and x_k are the end vertices of P.

Definition (Cycle)**.** *If $P = (x_0, x_1, \dots, x_{k-1})$ is a path and $k \geq 3$, then $C = P + \{x_{k-1}, x_0\}$ is called a cycle. Its length is k and we denoted it by C^k.*

Definition (Girth and circumference)**.** *Let G be a graph.*

(a) The minimal length of a cycle in G is the girth *(german: Taillenweite) $\mathrm{g}(G)$ of G.*

(b) The maximal length of a cycle in G is the circumference *$\mathrm{c}(G)$ of G.*

If G has no cycle at all then $\mathrm{g}(G) = \infty$ and $\mathrm{c}(G) = 0$.

Definition (Chord)**.** *Let $C^k = x_0x_1 \dots x_{k-1}$ be a cycle in a graph G. An edge $\{x_i, x_j\}$ with $1 \leq i \neq j \leq k-1$ joining two vertices of C^k such that $\{x_i, x_j\} \notin E(C^k)$ is called a* chord*. An induced cycle in G is a cycle without chords.* add pic

Proposition 1.1. *Every graph contains a path of length $\delta(G)$ and a cycle of length $\delta(G) + 1$, provided that $\delta(G) \geq 2$.*

Proof. Homework: Consider longest path □ add pic

Definition (Distance and diameter)**.** *Let G be a graph.*

(a) The distance *of two vertices $x, y \in V(G)$ is the length of the shortest x-y-path denoted by $\mathrm{dist}_G(x, y)$. Set $\mathrm{dist}_G(x, y) = \infty$ if there is no x-y-path in G.*

(b) The diameter of G is defined as

$$\mathrm{diam}(G) := \max_{x,y \in V(G)} \mathrm{dist}_G(x, y).$$

Proposition 1.2. *Every graph containing a cycle satisfies*

$$\mathrm{g}(G) \leq 2\,\mathrm{diam}(G) + 1.$$

Proof. Let C be a shortest cycle in G. If add pic

$$\mathrm{g}(G) \geq 2\,\mathrm{diam}(G) + 2,$$

then there exist $x, y \in V(C)$ such that

$$\mathrm{dist}_C(x, y) \geq \mathrm{diam}(G) + 1.$$

In G the condition $\mathrm{dist}_G(x, y) \leq \mathrm{diam}(G)$ holds, so any shortest path P between x, y is not a subgraph of C. Thus P contains a C-path $x'Py'$. Use $x'Py'$ and the shortest x'-y'-path in C to construct a cycle C' strictly shorter than C ↯. □

04.02.2016